Sahand University of Technology
Advanced Numerical Analysis
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1. Write a MATLAB code to evaluate cos(kx) using the three examples given in the stoer's book. Show your results for different values of k and x, especially for the case |kx|~1.

2. Consider the function $f(x) = (e^x - 1)/x = \sum_{i=0}^{\infty} x^i/(i+1)!$, which arises in various applications. The obvious way to evaluate f is via the algorithm

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% Algorithm 1.
if x = 0
  f = 1
else
  f = (e^x - 1)/x
end
```

This algorithm suffers severe cancellation for $|x| \ll 1$, causing it to produce an inaccurate answer (0 instead of 1, if x is small enough) . Give an alternative.

3. In error analysis it is sometimes convenient to bound $\widetilde{E}_{\mathrm{rel}}(\widehat{x}) = |x-\widehat{x}|/|\widehat{x}|$ instead of $E_{\mathrm{rel}}(\widehat{x}) = |x-\widehat{x}|/|x|$. Obtain inequalities between $E_{\mathrm{rel}}(\widehat{x})$ and $\widetilde{E}_{\mathrm{rel}}(\widehat{x})$.

4. Give stable formulae for computing the square root $x + iy$ of a complex number $a + ib$.

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5. Show how to rewrite the following expressions to avoid cancellation for the indicated arguments.

1. $\sqrt{x+1} - 1,\ x \approx 0.$
2. $\sin x - \sin y,\ x \approx y.$
3. $x^2 - y^2,\ x \approx y.$
4. $(1 - \cos x)/\sin x,\ x \approx 0.$
5. $c = (a^2 + b^2 - 2ab\cos\theta)^{1/2},\ a \approx b,\ |\theta| \ll 1.$

6. (Kahan, Muller, [781, 1989], Francois and Muller [406, 1991]) Consider the recurrence

$$x_{k+1} = 111 - (1130 - 3000/x_{k-1})/x_k, \qquad x_0 = 11/2, \quad x_1 = 61/11.$$

In exact arithmetic the x_k form a monotonically increasing sequence that converges to 6. Implement the recurrence on your computer or pocket calculator and compare the computed x_{34} with the true value 5.998 (to four correct significant figures). Explain what you see.

7. In the course of solving $ax^2 - 2bx + c = 0$ for x, the expression $\sqrt{b^2 - ac}$ must be computed. Can the true value of $b^2 - ac$ be nonnegative and yet its computed value be negative?

8. Two requirements that we might ask of a routine for computing $\sqrt{x}$ in floating point arithmetic are that the identities $\sqrt{x^2} = |x|$ and $(\sqrt{x})^2 = |x|$ be satisfied. Which, if either, of these is a reasonable requirement?

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9. Are there any floating point values of x and y (excepting values both 0, or so huge or tiny to cause overflow or underflow) for which the computed value of $x/\sqrt{x^2+y^2}$ exceeds 1?

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10. The standard met hod for evaluating a polynomial

$$p(x) = a0 + a_1 x + \dots + a_n x^n$$

is Horner's method (also known as Horner's rule and nested multiplication), which consists of the following recurrence:

$qn(x) = a_n$
for $i = n - 1: -1:0$
$\quad qi(x) = x q_{i+1}(x) + a_i$
end
$p(x) = q0(x)$

The cost is $2n$ flops, which is n less than the more obvious method of evaluation that explicitly forms powers of x. Give a bound for the round off error of evaluation with Horner's rule. How we can evaluate derivatives of a polynomial using Horner's rule?

11. Using 3-digit floating-point arithmetic, apply the classical Gram-Schmidt algorithm to the set

$$\mathbf{x}_1 = \begin{pmatrix} 1 \\ 10^{-3} \\ 10^{-3} \end{pmatrix}, \quad \mathbf{x}_2 = \begin{pmatrix} 1 \\ 10^{-3} \\ 0 \end{pmatrix}, \quad \mathbf{x}_3 = \begin{pmatrix} 1 \\ 0 \\ 10^{-3} \end{pmatrix}.$$

show that the resulted vectors are not orthogonal. Explain why? Use the modified Gram-Schmidt algorithm to obtain a set of orthogonal vectors.

Modified Gram–Schmidt Algorithm

For a linearly independent set $\{\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n\} \subset \mathcal{C}^{m\times 1}$, the Gram–Schmidt sequence can be alternately described as

$$\mathbf{u}_k = \frac{\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{x}_k}{\|\mathbf{E}_k \cdots \mathbf{E}_2\mathbf{E}_1\mathbf{x}_k\|} \text{ with } \mathbf{E}_1 = \mathbf{I},\ \mathbf{E}_i = \mathbf{I} - \mathbf{u}_{i-1}\mathbf{u}_{i-1}^* \text{ for } i > 1,$$

and this sequence is generated by the following algorithm.

For $k = 1$: $\quad \mathbf{u}_1 \leftarrow \mathbf{x}_1 / \|\mathbf{x}_1\|$ and $\mathbf{u}_j \leftarrow \mathbf{x}_j$ for $j = 2, 3, \ldots, n$

For $k > 1$: $\quad \mathbf{u}_j \leftarrow \mathbf{E}_k\mathbf{u}_j = \mathbf{u}_j - (\mathbf{u}_{k-1}^*\mathbf{u}_j)\,\mathbf{u}_{k-1}$ for $j = k, k+1, \ldots, n$

$\quad \mathbf{u}_k \leftarrow \mathbf{u}_k / \|\mathbf{u}_k\|$